A line bundle on an algebraic space which is not Zariski locally trivial

January 23, 2026

Let X be a smooth projective threefold over $\mathbf{C}$ with $\mathrm{Pic}(X) = \mathbf{Z}$ and with a free action of a finite cyclic group $G \neq 1$. Assume there is a smooth rational curve $\mathbf{P}^1_k \cong C \subset X$ such that $\mathcal{N}^\vee_{C/X}$ is an ample vector bundle on C and such that $g(C) \cap h(C) = \emptyset$ for $g \neq h$ in G. We will show later that such an X exists.

Let $f : X \to Y$ be the normal, proper algebraic space variety over $\mathbf{C}$ obtained by contracting each curve gC for $g \in G$ to a (separate) point and then normalizing if necessary. This exists by Artin's theorem on existence of contractions [Art70] and the simpler criterion for existence due to [Maz75]. By functoriality of this construction and the fact that the action of G preserves the closed subscheme $\cup_{g \in G} g(C)$, the action of G on X descends to an action of G on Y which we see is free. Let $\pi : Y \to Y/G = Z$ be the quotient. Since π is finite étale, we see Z is also a proper normal algebraic space variety over $\mathbf{C}$. The union $\bigcup_{g \in G} g(C) \subset X$ maps to a single point $z \in Z$. Also, choosing a root of unity, we may view $Y \to Z$ as a μ_n-torsor, $n = |G|$. Let $\mathcal{L}$ be the corresponding n-torsion line bundle.

Claim: There is no open algebraic subspace $U \subset Z$ containing z such that $\mathcal{L}|_U \cong \mathcal{O}_U$.

Suppose U is such an open. Let $F = Z \setminus U$ be the complement. We claim that $\dim(F) \leq 1$. If not, choose an irreducible component W_0 of dimension 2. Then the preimage of W_0 in X is a non-empty effective Cartier divisor disjoint from all of the curves gC, but every non-empty effective Cartier divisor on X is ample, so this is impossible. Therefore, $\mathcal{O}_Z(U) = \mathbf{C}$. Since $\pi^{-1}(U) \to U$ is a μ_n-torsor for which the associated $\mathbf{G}_m$-torsor is trivial, the Kummer sequence shows there is $a \in H^0(U, \mathcal{O}_U)^\times = \mathbf{C}^\times$ such that $\pi^{-1}(U) = \underline{\mathrm{Spec}}(\mathcal{O}_U[T]/(T^n - a)$ over U. But since $a \in \mathbf{C}^\times$, this shows $Y \supset \pi^{-1}(U) = \coprod_{i=1}^n U$, which contradicts the fact that Y is irreducible.

An example of such an X is the quintic hypersurface

$$T_0^5 + T_1^5 + \cdots + T_4^5 - T_0 T_1 \cdots T_4 = 0$$

in $\mathbf{P}^4_{\mathbf{C}}$. The group $G = \mathbf{Z}/5$ acts freely on X by cyclically permuting the coordinates. We let $\zeta \neq -1$ be a 5^{th} root of -1 and for $C \subset X$ we take the line

$$(t : -t : u : \zeta u : 0), \qquad (t : u) \in \mathbf{P}^1.$$

It is observed in [CdlOvGvS12, Page 2] that this line is isolated in the Fano scheme of X. Here that is equivalent to having normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (the other possibilities for the normal bundle of a line in a quintic threefold are $\mathcal{O} \oplus \mathcal{O}(-2)$ and $\mathcal{O}(1) \oplus \mathcal{O}(-3)$). One checks directly that $gC \cap hC = \emptyset$ for $g \neq h$ in G, so this works.

References

[Art70] Michael Artin. Algebraization of formal moduli: II – existence of modifications. *Annals of Mathematics*, 91:88–135, 1970.

[CdlOvGvS12] Philip Candelas, Xenia de la Ossa, Bert van Geemen, and Duco van Straten. Lines on the dwork pencil of quintic threefolds, 2012.

[Maz75] Joseph Mazur. Conditions for the existence of contractions in the category of algebraic spaces. *Transactions of the American Mathematical Society*, 209:259–265, 1975.